Journal of Combinatorial Theory, Series A 117 (2010) 625–641

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

The necklace poset is a symmetric chain order

Kelly Kross Jordan [1]

Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA

ARTICLE INFO

Article history:
Received 8 January 2008
Available online 4 March 2010

Keywords:
Poset
Boolean lattice
Venn diagram
Symmetric chain decomposition
Bracketing
Necklace poset
Quotient poset

ABSTRACT

Let N_n denote the quotient poset of the Boolean lattice, B_n, under the relation equivalence under rotation. Griggs, Killian, and Savage proved that N_p is a symmetric chain order for prime p. In this paper, we settle the question posed in that paper, namely whether N_n is a symmetric chain order for all n. This paper provides an algorithm that produces a symmetric chain decomposition (or SCD). We accomplish this by modifying bracketing from Greene and Kleitman. This allows us to take appropriate "middles" of certain chains from the Greene–Kleitman SCD for B_n. We also prove additional properties of the resulting SCD and show that this settles a related conjecture.

Published by Elsevier Inc.

1. Introduction

In this paper, we prove that the necklace poset, N_n, in fact has a symmetric chain decomposition (SCD). In Section 2, we introduce some terms related to posets. We also give a description and proof of the Greene–Kleitman SCD for the Boolean lattice and define and discuss known properties of N_n. In Section 3, we introduce three lemmas without proof and use them to prove that N_n has an SCD. In Section 4, we introduce the idea of circular matchings and prove various properties of these matchings. In Section 5, we use circular matchings to prove the lemmas from Section 3. This completes the proof that N_n has an SCD. In Section 6, we modify the proof in Section 3 and use the modified proof to answer a related conjecture. Finally, in Section 7, we offer some open questions.

2. Symmetric chain decompositions in the Boolean lattice

We begin with some important definitions, following Anderson [1] and Engel [3]. A *chain* in a poset, $(P, <)$, is a totally ordered subset of P. The *length* of a chain is one less than its cardinality. In a poset, $(P, <)$, for some elements x and y of P, we say that x *covers* y if $x > y$ and there is

E-mail address: kelly.kross@yahoo.com.

[1] Research supported in part by NSF grant DMS-0072187.

0097-3165/$ – see front matter Published by Elsevier Inc.
doi:10.1016/j.jcta.2010.02.003

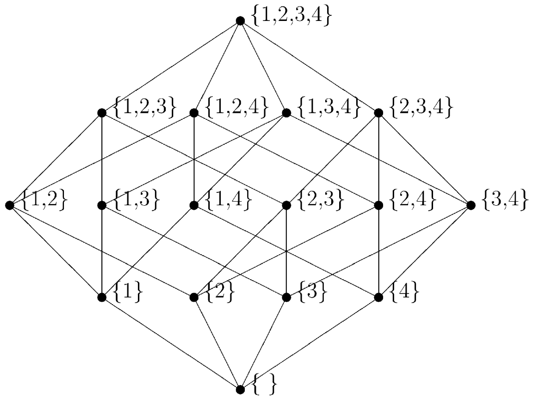

Fig. 1. The Hasse diagram for B_4.

no element z such that $x > z > y$. A *saturated chain* is a chain $x_1 < \cdots < x_k$ such that x_i covers x_{i-1} for each $i > 1$. If there is a unique element E in P such that $E \leqslant x$ for all $x \in P$, we say that E is the *zero element* of $(P, <)$. We say a poset is *ranked* if it has the property that for any $x < y$, all saturated chains from x to y have the same length. In a ranked poset $(P, <)$, we define the *rank*, $r(x)$, of an element to be the length of each chain from the zero element of the poset to x. For $x_i \in P$, the saturated chain $x_1 < x_2, <, \ldots, < x_k$ is a *symmetric chain* in P if

$$r(x_1) + r(x_k) = r(P)$$

where $r(P)$ is the maximum rank in P. A *symmetric chain decomposition* (or SCD) of P is a partition of P into symmetric chains $C_1, \ldots, C_k$. If a poset has an SCD, we say it is a *symmetric chain order*, (or SCO). Figs. 3 and 4 demonstrate two representations of an SCD for B_4.

In this paper, we are primarily interested in subposets and quotients of the *Boolean lattice*, B_n, which is the poset of subsets of the set $[n] = \{1, \ldots, n\}$ ordered by inclusion. Refer to Fig. 1 for the Hasse diagram of B_4. A chain in B_n consists of elements $A_i \in B_n$ with $A_1 \subset, \ldots, \subset A_k$. It is clear that B_n is a ranked poset, with rank function $r(A) := |A|$. The chain $A_1 < A_2 < \cdots < A_k$ is a symmetric chain in B_n if for $i = 1, \ldots, k-1$, we have $|A_{i+1}| = |A_i| + 1$, and $|A_1| + |A_k| = n$. There are several proofs of the fact that B_n is an SCO (see [2] and [4]).

Greene and Kleitman provide a particularly nice construction of an SCD for B_n (see [4]). To a set $A \in B_n$ with $A = \{x_1, \ldots, x_k\}$, we associate a sequence $\hat{A}$ of zeros and ones of length n, so that $\hat{A}$ has a one in position i if and only if $i \in A$. For example, in B_7, the set $\{2, 3, 6\}$ corresponds to the sequence 0110010. Refer to Fig. 2 for the Hasse diagram of B_4 represented by $\{0, 1\}$ sequences. In this paper, elements of B_n will be primarily represented by and referred to by these sequences. Using these $\{0, 1\}$ sequences, we then perform a procedure equivalent to matching and closing parentheses with "(" represented by a zero and ")" represented by a one. This procedure is commonly referred to as bracketing or parenthesis matching. Formally, starting at the left, when we encounter a zero, it becomes (possibly temporarily) unmatched. When a one is encountered, it is matched to the rightmost unmatched zero, and this zero is now matched as well. If there are currently no unmatched zeros, then this one is unmatched. We continue in this manner until we reach the end of the sequence. We should now have three sets associated with the given sequence x: The set of positions of unmatched zeros, $U_0(x)$, the set of positions of unmatched ones, $U_1(x)$, and finally, the set of matchings, $M(x) := \{(a, b)\text{: a zero in position } a \text{ is matched to a one in position } b\}$. For example, if $x = 1011011100010110$, then the parenthesis version is)())()))((()())(, and when we perform the matching, we get:

$$U_0(x) = \{9, 16\}$$
$$U_1(x) = \{1, 4, 7, 8\}$$
$$M(x) = \big\{(2, 3), (5, 6), (10, 15), (11, 12), (13, 14)\big\}.$$

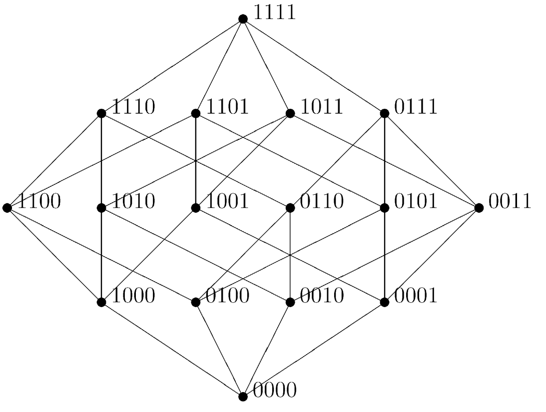

Fig. 2. The Hasse diagram for B_4, represented by $\{0,1\}$ sequences.

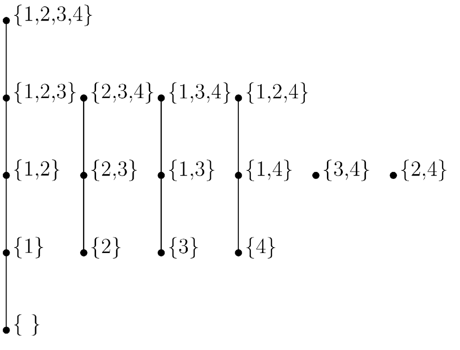

Fig. 3. The Greene–Kleitman SCD for B_4.

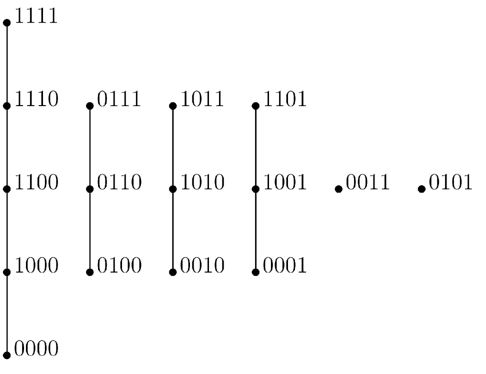

Fig. 4. The Greene–Kleitman SCD for B_4, represented by $\{0,1\}$ sequences.

We should establish an important fact about these sets. If $a \in U_1(x)$ and $b \in U_0(x)$, then $a < b$. That is, all unmatched ones precede all unmatched zeroes. (If $b < a$, then the zero in position b was encountered before the one in position a. So, position b consisted of an unmatched zero when the one in position a was encountered, and the one in position a would not have become unmatched.)

We next introduce a function τ which acts on the $\{0,1\}$ sequences by changing the leftmost unmatched zero to a one. The function τ is defined on all $x \in B_n$ such that $U_0(x) \neq \emptyset$. By the fact above, we observe that:

$$U_0\big(\tau(x)\big) = U_0(x) \setminus \{i\}$$
$$U_1\big(\tau(x)\big) = U_1(x) \cup \{i\}$$
$$M\big(\tau(x)\big) = M(x)$$

where $i = \min(U_0(x))$. We also define τ^{-1} which changes the rightmost unmatched one to a zero. It is defined on all $x \in B_n$ such that $U_1(x) \neq \emptyset$. We observe that:

$$U_0\big(\tau^{-1}(x)\big) = U_0(x) \cup \{i\}$$
$$U_1\big(\tau^{-1}(x)\big) = U_1(x) \setminus \{i\}$$
$$M\big(\tau^{-1}(x)\big) = M(x)$$

where $i = \max(U_1(x))$. From the observations above, we conclude that for $x \in B_n$ such that $U_0(x) \neq \emptyset$, we have that $\tau^{-1}(\tau(x)) = x$. Similarly, for $x \in B_n$ such that $U_1(x) \neq \emptyset$, we have $\tau(\tau^{-1}(x)) = x$. Thus, $\tau(x)$ is one-to-one.

The following theorem gives a construction of the Greene–Kleitman SCD for B_n.

Theorem 1. *(See Greene and Kleitman [4].) For a x in B_n with $|U_0(x))| = k$, let $C_x = \{x, \tau(x), \tau^2(x), \dots, \tau^k(x)\}$. The following is a symmetric chain decomposition of B_n:*

$$S = \big\{C_x \,\big|\, x \in B_n,\ U_1(x) = \emptyset\big\}.$$

Proof. Using the facts above about τ, we construct the chains of the Greene–Kleitman SCD for B_n. For x in B_n with $U_1(x) = \emptyset$ and $|U_0(x)| = k$, let $C_x = \{x, \tau(x), \tau^2(x), \dots, \tau^k(x)\}$ be a chain in the decomposition. We need to show that C_x is in fact symmetric. Note that

$$\begin{aligned}|x| + \big|\tau^k(x)\big| &= \big|M(x)\big| + \big|U_1(x)\big| + \big|M\big(\tau^k(x)\big)\big| + \big|U_1\big(\tau^k(x)\big)\big| \\ &= 2\big|M(x)\big| + k \\ &= n\end{aligned}$$

because $2|M(x)| + k$ is simply the total number of zeros and ones in x. Any matching accounts for two positions, and any unmatched position in x is an unmatched zero. So, C_x is symmetric. The fact that $\tau(x)$ is one-to-one proves that the chains in S are disjoint. Further, for $y \in B_n$ with $k = \max\{i \mid U_0(\tau^{-i}(y)) > 0\}$, let $x = \tau^{-k-1}(y)$. By our choice of k, we see that $U_1(x) = \emptyset$. So the chain C_x is in S. Note also that $\tau^{k+1}(x) = y$, so that $y \in C_x$. Since x was chosen arbitrarily, S is a partition of B_n. □

We now define several additional properties of posets, in the manner of [1] and [3]. Let P be a ranked poset with maximum rank M where $P_k = \{x \in P\colon\ rank(x) = k\}$. Then, P is *rank-symmetric* if, given $k = 0, 1, 2, \dots, M$, we have $|P_k| = |P_{M-k}|$. Further, P is *rank-unimodal* if there exists j such that $|P_0| \leqslant |P_1| \leqslant \dots \leqslant |P_j|$ and $|P_j| \geqslant |P_{j+1} \geqslant \dots \geqslant |P_M|$. The *$k$ middle levels* of a poset are composed of the elements with the k middle ranks, where k must have opposite cardinality of the maximal rank of the poset. For example, for a ranked poset with maximal rank n, where n is even, the 3 middle levels would be the elements of rank $\frac{n}{2} - 1$, $\frac{n}{2}$, and $\frac{n}{2} + 1$. An *antichain* is a set of pairwise uncomparable elements of a poset. The poset P is *strongly Sperner* if, for all $k = 1, 2, \dots, M + 1$, the union of the k middle levels of P is a union of k antichains of maximum size. A poset is *Peck* if it is rank-symmetric, rank-unimodal, and strongly Sperner. Finally, given a group G of automorphisms of a poset P, the set of orbits of the automorphism form a *quotient of P under G* (or P/G) ordered in the following way: For orbits of G, A and B, we have $A \leqslant_{P/G} B$ if and only if there are $a \in A$ and $b \in B$ such that $a \leqslant_P b$. It is simple to see that this structure is a poset.

We are now ready to define necklaces and the necklace poset.

First, we define σ, the function that rotates an element of B_n. For $x \in B_n$, with $x = (x_1, x_2, \dots, x_n)$, ($x_i \in \{0, 1\}, i = 1, 2, \dots, n$), define

$$\sigma(x) = (x_n, x_1, \dots, x_{n-1}).$$

For $x, y \in B_n$, y we say *is a rotation of* x (or $y \sim x$) if for some k, $y = \sigma^k(x)$. It is clear that "$\sim$" is an equivalence relation on B_n.

Definition 2. The *necklace poset*, N_n is the quotient poset of B_n under the relation $\sim$, where for $X, Y \in N_n$, $X \leqslant Y$ if there exist $x \in X$ and $y \in Y$ ($x, y \in B_n$) with $x \subseteq y$ [7].

Refer to Fig. 5 for the Hasse diagram of N_6.

We now discuss N_n in relation to the previously defined properties. By definition, the necklace poset is a quotient of the Boolean lattice, because its elements are orbits of the elements of B_n under the rotation automorphism. Stanley showed that any quotient of the Boolean lattice is a Peck poset, using the fact that it is unitary, a property we will not define here.

Theorem 3. *(See Stanley [11].) If P is a unitary Peck poset, then P/G is Peck.*

Stanley also proved that B_n is unitary Peck for all n (see [11]). Therefore, N_n satisfies the properties of rank symmetry, rank unimodality, and is strongly Sperner.

Griggs (see [8]) showed that the LYM property (which we will not define here), together with rank-symmetry and rank-unimodality, implies that a poset has a symmetric chain decomposition. For prime p, it may be easily verified that N_p satisfies the LYM property, and therefore has an SCD. It is not known whether N_n has the LYM property in the general case. However, the fact that the general N_n is Peck lent some support that it had an SCD.

In a paper on symmetric Venn diagrams, Griggs, Killian and Savage (see [7]) gave an elegant explicit construction of an SCD for N_p, with p prime. This SCD has an additional property, the chain cover property, which we will discuss in Section 5. They used the idea of bracketing from the Greene–Kleitman SCD for B_n, which we also use in this paper. They also used the idea of block codes to choose a representative in B_n for each element of N_n. Denote by R_n this subposet of representatives. (Note that $R_n \subset B_n$.)

Theorem 4. *(See Griggs, Killian, and Savage [7].) If n is prime, R_n has a symmetric chain decomposition with the chain cover property.*

In the same paper, Griggs, Killian, and Savage posed the question of whether N_n is a symmetric chain order for all n, the question we answer here. Jiang and Savage [10] applied some of the methods in [7] to the case of composite n. They were able to narrow the problem to that of finding an SCD for the elements of N_n with periodic block code. It is possible to find SCDs for the elements of N_n with periodic block codes for n up to 16. So, there exist SCDs for N_n with $n \leqslant 16$.

3. The necklace poset is an SCO

In this section, we prove that N_n has an SCD in the general case. The proof that N_n has an SCD utilizes three lemmas. The lemmas demonstrate that we can perform certain operations on the Greene–Kleitman SCD for B_n while preserving the property that each chain is symmetric. These operations allow us to remove all but one representative from each equivalence class in N_n, leaving a symmetric chain decomposition for N_n. In this section, we assume the lemmas and use them to prove the following theorem. We will prove the lemmas in Section 5.

Theorem 5. *For all positive integers n, N_n is a symmetric chain order.*

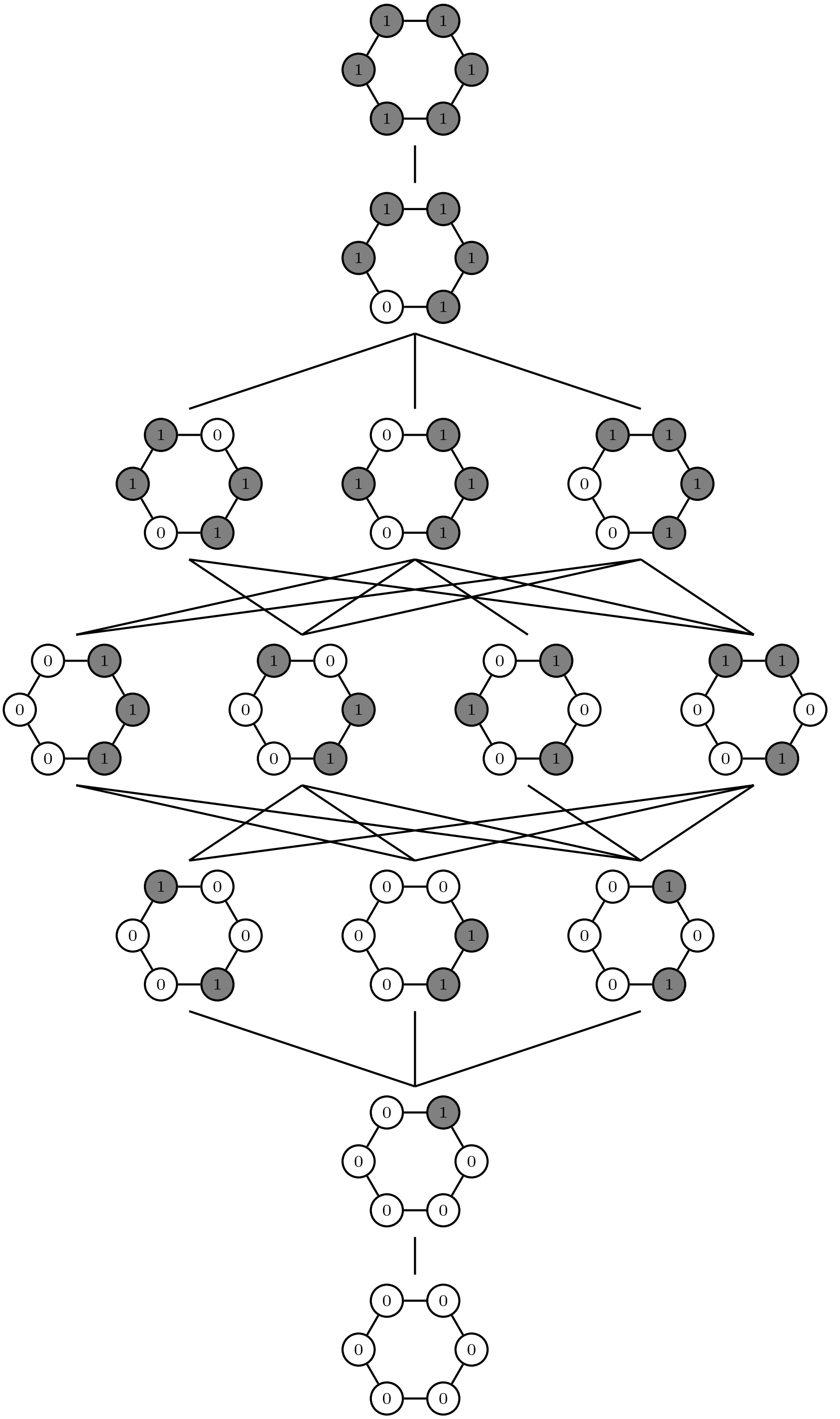

Fig. 5. The Hasse diagram for N_6.

111111										
111110	011111	101111	110111	111011	111101					
111100	011110	**101110**	**110110**	**111010**	111001	001111	100111	110011	010111	101011
111000	011100	**101100**	**110100**	**110010**	110001	001110	100110	100011	010110	**101010**
110000	011000	**101000**	**100100**	**100010**	100001	001100	000110	000011	010100	001010
100000	010000	001000	000100	000010	000001					
000000										

110101	101101	011011	011101					
100101	101001	011010	011001	000111	001101	010011	010101	001011
000101	001001	010010	010001					

Fig. 6. SCD for B_6 with members of M_n in bold.

111111					
111110					
111100	101110	$\longleftrightarrow$	111010	110110	
111000	101100	$\longleftrightarrow$	110010	110100	101010
110000	101000	$\longleftrightarrow$	100010	100100	
100000					
000000					

Fig. 7. SCD for M_6 with duplicate representatives of N_6 members indicated.

Proof. We define a set M_n, consisting of $x \in B_n$ such that x achieves the maximum number of unmatched ones over all rotations, that is,

$$M_n = \{y \in B_n\colon\ |U_1(y)| = \max\{|U_1(\sigma^k(y)|\colon\ k = 1, 2, \ldots, n\}\}.$$

We use the set M_n in the first two lemmas. Fig. 6 shows an SCD of B_6 with the member of M_n in bold.

Lemma 6. *Let $x \in M_n$. Then, if $|x| < \frac{n}{2}$,*

$$\tau^i(x) \in M_n, \quad 1 \leqslant i \leqslant n - 2|x|$$

and if $|x| > \frac{n}{2}$,

$$\tau^{-i}(x) \in M_n, \quad 1 \leqslant i \leqslant 2|x| - n.$$

That is, if $x \in M_n$ and C is the chain containing x in the Greene–Kleitman SCD of B_n, all of the elements of the smallest symmetric "sub-chain" of C that contains x are also in M_n.

This lemma allows us to remove all of the elements of B_n that are not also in M_n. Note that the resulting chains still contain at least one representative of every element of N_n. We will refer to the remaining chains as the SCD for M_n. Refer to Fig. 7 for an SCD of M_6 with duplicate representatives of N_6 elements indicated. The next two lemmas allow us to eliminate remaining duplicate representatives of elements of N_n. Refer to Fig. 8 for an illustration of what is going on in Lemma 7.

Lemma 7. *Let $x, y \in M_n$ with $x \sim y$.*
If $|x| \geqslant \frac{n}{2}$, then $\tau(x) \sim \tau(y)$ or $\{\tau(x), \tau(y)\} \cap M_n = \emptyset$.
If $|x| \leqslant \frac{n}{2}$, then $\tau^{-1}(x) \sim \tau^{-1}(y)$ or $\{\tau^{-1}(x), \tau^{-1}(y)\} \cap M_n = \emptyset$.

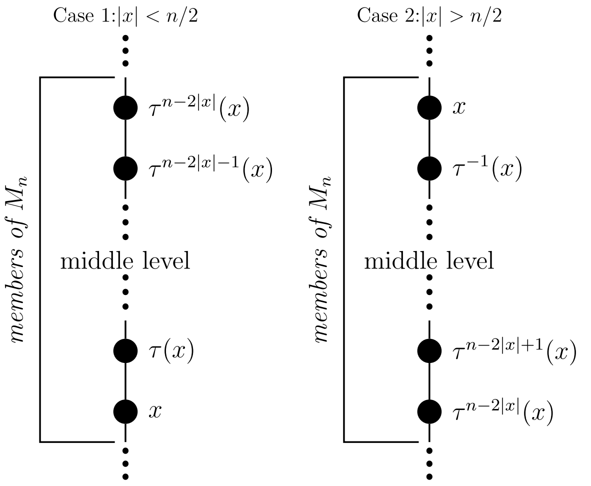

Fig. 8. Lemma 7 illustrated.

Lemma 8. *Let $x, y \in B_n$ with $|x| = |y| = k < \frac{n}{2}$. Then,*

$$x \sim y \quad \Leftrightarrow \quad \tau^{n-2k}(x) \sim \tau^{n-2k}(y).$$

In the rest of the proof, we describe an algorithm that produces an SCD for N_n from the SCD for M_n. In this algorithm, the kth iteration produces an SCD for a subset D^k of B_n. We always preserve the property that each necklace element has at least one representative in D^k, and $D^k \subsetneq D^{k-1}$. The algorithm terminates if there are no distinct $x, y \in D^k$ with $x \sim y$. Otherwise, another iteration produces D^{k+1}.

Initially, let C_x^0 be the chain in the Greene–Kleitman SCD for B_n, restricted to M_n, that contains x. Also, let $D^0 = M_n$. Before step $j+1$ in the iteration, C_x^j is the chain containing x, and D^j is the set of elements of B_n remaining in the poset. For iteration $j+1$ let $x, y \in D^j$ be distinct with $x \sim y$, $|x| = k$. If there are no such x and y, then we have an SCD for N_n. Without loss of generality, suppose that $|x| \leqslant \frac{n}{2}$. Otherwise, by Lemma 8, we can choose $\tau^{n-2k}(x)$ and $\tau^{n-2k}(y)$. We also assume that C_x^j is at least as long as C_y^j. By repeated application of Lemma 7, we get that for all $i \geqslant 0$ with $\tau^{-i}(y) \in M_n$, $\tau^{-i}(x) \sim \tau^{-i}(y)$. This corresponds to the "bottom tail" of C_y^j. Define the "bottom tail" by:

$$T_b^j := \{\tau^{-i}(y) \mid i \geqslant 0,\ \tau^{-i}(y) \in M_n\}.$$

Using Lemma 8, we get that $\tau^{n-2k}(x) \in M_n$. Then, applying Lemma 7 repeatedly, we get that for all $i \geqslant 0$ with $\tau^{n-2k+i}(y) \in M_n$, $\tau^{n-2k+i}(x) \sim \tau^{n-2k+i}(y)$. This corresponds to the "top tail" of C_y^j. Define the "top tail" by:

$$T_t^j := \{\tau^{n-2k+i}(y) \mid i \geqslant 0,\ \tau^{n-2k+i}(y) \in M_n\}.$$

We then remove the tails of C_y^j. That is, we set

$$C_*^{j+1} = C_y^j \setminus (T_b^j \cup T_t^j)$$
$$D^{j+1} := D_j \setminus (T_b^j \cup T_t^j).$$

Fig. 9 represents this visually.

The new chain, C_*^{j+1} is symmetric, and we have only removed members which were rotations of members of the chain containing x. For $z \in D^{j+1} \setminus C_y^j$, set $C_z^{j+1} := C_z^j$. Also, for $z \in C_*^{j+1}$, let

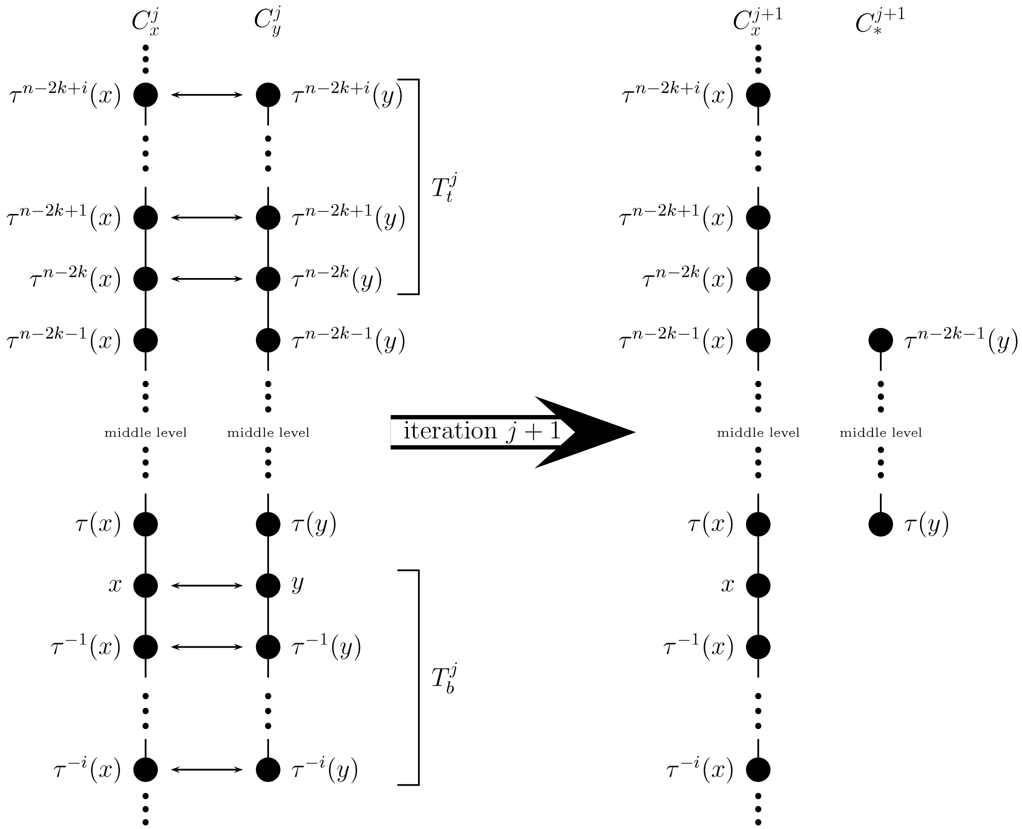

Fig. 9. Removing duplicate representatives of elements of M_n.

$C_z^{j+1} := C_*^{j+1}$. The set D^{j+1} has at least one fewer duplicate representative than D^j, and the following is an SCD for D^{j+1}:

$$\bigcup_{z \in D^{j+1}} C_z^{j+1}.$$

So given the three lemmas, the theorem holds. $\square$

For an example of the finished output, refer to Fig. 10.

4. Circular matchings

To prove the lemmas, we introduce the idea of circular matching, which remains structurally unchanged under rotation. Intuitively, we arrange the string of zeros and ones in a circle and match them in the same manner Greene and Kleitman did in a straight line. Formally, we must pick a starting position, although we will later prove that the end result does not depend on this starting position. This starting position, together with the necklace element, corresponds to an element x of B_n. We first perform the normal Greene and Kleitman parenthesis matching process, forming sets $U_0(x)$, $U_1(x)$, and $M(x)$. Then, we iteratively form the sets $CU_0(x)$, $CU_1(x)$, and $CM(x)$, the set of cir-

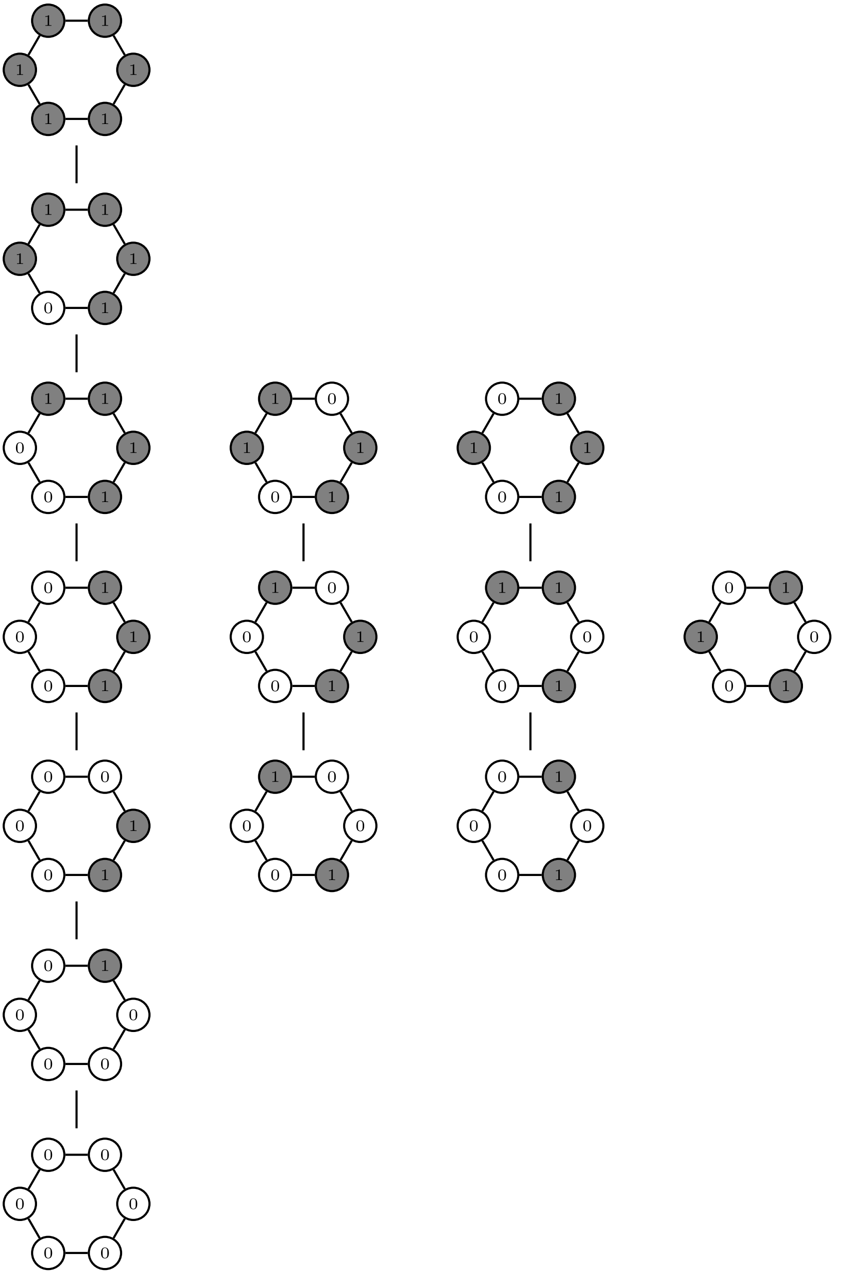

Fig. 10. SCD for N_6.

cularly unmatched zeros, circularly unmatched ones, and circular matchings, respectively. Start with $CU_0^0(x) = U_0(x)$, $CU_1^0(x) = U_1(x)$, and $CM^0(x) = M(x)$. At step i, let

$$a := \max\bigl(CU_0^i(x)\bigr)$$
$$b := \min\bigl(CU_1^i(x)\bigr).$$

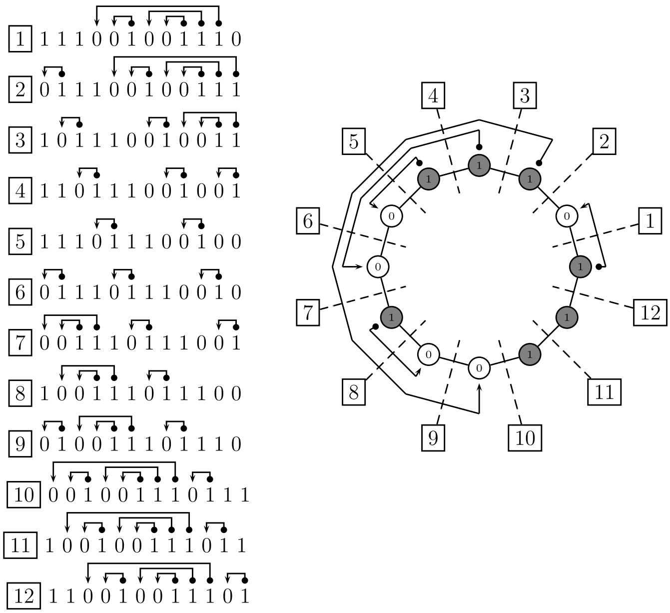

Fig. 11. B_{12} representations of an element of N_{12}, with cuts.

Note here that $b < a$. Then define,

$$CM^{i+1}(x) := CM^i(x) \cup \{(a, b)\}$$
$$CU_0^{i+1}(x) := CU_0^i(x) \setminus \{a\}$$
$$CU_1^{i+1}(x) := CU_1^i(x) \setminus \{b\}.$$

Continue until $CU_0^i(x) = \emptyset$ or $CU_1^i(x) = \emptyset$. At this point, set

$$CM(x) := CM^i(x)$$
$$CU_0(x) := CU_0^i(x)$$
$$CU_1(x) := CU_1^i(x).$$

We next establish some properties of these sets. There is an intuitive order of the matchings with the relation "inside of." These matchings can always be represented visually non-intersecting directed arcs joining the specified zeros and ones arranged on the boundary of a disc. Refer to the right side of Fig. 11 for an example of these arcs. Intuitively, it is clear that this can always be done. However, for the reader who desires a more rigorous proof, we now formalize this partial ordering. Define:

$$(a, b)^* := \begin{cases} I(a, b) & \text{if } a < b \\ [0, b) \cup (a, n] & \text{if } a > b \end{cases}.$$

We use the notation $I(a, b)$ to refer to the open interval (a, b) in order to avoid confusion with our notation for the circular matching (a, b).

Proposition 9. *For $x \in B_n$, the set $CM(x)$ with the order $(a_1, b_1) <_m (a_2, b_2)$ if $(a_1, b_1)^* \subset (a_2, b_2)^*$, is a partial order such that (a_1, b_1) and (a_2, b_2) are incomparable if and only if $(a_1, b_1)^* \cap (a_2, b_2)^* = \emptyset$.*

Proof. First, it is clear that the above induces a partial order on $CM(x)$. Next, if $(a_1, b_1)^* \cap (a_2, b_2)^* = \emptyset$, then clearly (a_1, b_1) and (a_2, b_2) are incomparable. We prove the converse by cases. Assume $(a_1, b_1)^* \cap (a_2, b_2)^* \neq \emptyset$, and we assume, without loss of generality, that $a_1 < a_2$.

Case 1: $\boldsymbol{b_1 < b_2 < a_1 < a_2}$**.** Both matchings are in $CM(x) \setminus M(x)$. So at the step that the circular matching (a_1, b_1) was added to $CM^{i+1}(x)$, $a_1 = min(CU_0^i(x))$ and $b_1 = max(CU_1^i(x))$. This means that the circular matching (a_2, b_2) had to have been added first. But using the same reasoning, this also couldn't have happened. So, this case simply never happens.

Case 2: $\boldsymbol{b_2 < b_1 < a_1 < a_2}$**.** Here, $(a_2, b_2)^* = [0, b_2) \cup (a_2, n] \subset [0, b_1) \cup (a_1, n] = (a_1, b_1)^*$, so that $(a_2, b_2) <_m (a_1, b_1)$.

Case 3: $\boldsymbol{a_1 < b_1 < b_2 < a_2}$**.** Here, $(a_1, b_1)^* = I(a_1, b_1) \subset [0, b_2) \cup I(a_2, n] = (a_2, b_2)^*$, so that $(a_1, b_1) <_m (a_2, b_2)$.

Case 4: $\boldsymbol{a_1 < b_2 < b_1 < a_2}$**.** Here, in the initial Greene–Kleitman matching phase, b_2 was encountered when a_1 was an unmatched zero, so a_1 would have been matched to b_2 instead. This case never happens.

Case 5: $\boldsymbol{a_1 < a_2 < b_1 < b_2}$**.** Similar to Case 4, in the initial Greene–Kleitman matching phase, a_1 and a_2 were both unmatched zeros when b_1 was encountered. Since $a_2 > a_1$, b_1 would have been matched to a_2 instead. This case never happens.

Case 6: $\boldsymbol{a_1 < a_2 < b_2 < b_1}$**.** Here, $(a_2, b_2)^* = I(a_2, b_2) \subset I(a_1, b_1) = (a_1, b_1)^*$, so that $(a_2, b_2) <_m (a_1, b_1)$.

Case 7: $\boldsymbol{b_1 < a_1 < a_2 < b_2}$**.** Here, $(a_2, b_2)^* = I(a_2, b_2) \subset [0, b_1) \cup I(a_1, n] = (a_1, b_1)^*$, so that $(a_2, b_2) <_m (a_1, b_1)$.

Case 8: $\boldsymbol{b_2 < a_1 < a_2 < b_1}$**.** During the initial Greene–Kleitman matching phase, a_1 and a_2 were both unmatched zeros when b_1 was encountered. Since $a_2 > a_1$, b_1 would have been matched to a_2 instead. This case never happens.

Case 9: $\boldsymbol{b_1 < a_1 < b_2 < a_2}$**.** During the initial Greene–Kleitman matching phase, a_1 was an unmatched zero when b_2 was encountered, so a_1 would have been matched to b_2. This case never occurs.

Case 10: $\boldsymbol{b_2 < a_1 < b_1 < a_2}$**.** Here, $(a_1, b_1)^* = I(a_1, b_1)$ is disjoint from $[0, b_2) \cup (a_2, n] = (a_2, b_2)^*$, which contradicts our assumption.

Case 11: $\boldsymbol{a_1 < b_1 < a_2 < b_2}$**.** Here, $(a_1, b_1)^* = I(a_1, b_1)$ is disjoint from $I(a_2, b_2) = (a_2, b_2)^*$, which contradicts our assumption.

Case 12: $\boldsymbol{a_1 < b_2 < a_2 < b_1}$**.** During the initial Greene–Kleitman matching phase, a_1 was an unmatched zero when b_2 was encountered, so a_1 would have been matched to b_2. This case never occurs. □

This proposition verifies that the circular matching procedure is equivalent to the parenthesis matching and closing process. Thinking of circular matching in this manner, we first match all of the minimal elements of the poset in Proposition 9. Then, we remove them from the poset and repeat the process. These matchings are illustrated in Fig. 11 by directed arcs.

We have previously alluded to the fact that the sets $CM(x)$, $CU_0(x)$, and $CU_1(x)$ are "structurally unchanged" under rotation. In fact, the sets above simply rotate as we rotate x, as it appears in Fig. 11. To make notation simpler, in the rest of the paper all addition will be performed modulo n. We now prove the following proposition:

Proposition 10. *Let $x \in B_n$. Then, for $i \in \{0, \dots, n-1\}$,*

$$\begin{aligned} CM\big(\sigma^i(x)\big) &= \big\{(a+i, b+i)\colon (a,b) \in CM(x)\big\} \\ CU_0\big(\sigma^i(x)\big) &= \big\{a+i\colon a \in CU_0(x)\big\} \\ CU_1\big(\sigma^i(x)\big) &= \big\{a+i\colon a \in CU_1(x)\big\}. \end{aligned}$$

Proof. First note that it is enough to show that if $(a,b) \in CM(x)$, then $(a+1, b+1) \in CM(\sigma(x))$. We can prove this by using the fact that the circular matchings are equivalent to the procedure of closing parentheses. In other words, we first close and remove the sequences that read 01 (moving clockwise), iterating this process until there are no more such sequences. In the case of linear Greene–Kleitman matching, this is when the sequence consists of all of the ones followed by all of the zeros. In the circular case, this is when the necklace consists of either all ones or all zeros or is the empty necklace. It is easy to see that, in the circular case, if there is a sequence 01 starting at position a (moving clockwise) in x, then $(a, a+1 \pmod n) \in CM(x)$. It is clear that if such a sequence is in x, there will be a sequence 01 starting in position $a+1$ in $\sigma(x)$. So then $(a+1, a+2) \in CM(\sigma(x))$. We can then remove the sequences corresponding to these matchings. The rest follows by induction on the size of the necklace. □

Proposition 11. *The structure of the poset of circular matchings is preserved under rotation. That is, if $(a_1, b_1) <_m (a_2, b_2)$, then $(a_1+1, b_1+1) <_m (a_2+1, b_2+1)$.*

Proof. This follows immediately from Propositions 9 and 10. □

Proposition 12. *Let $X \in N_n$. For any representative $x \in B_n$ of X, $k \in \{1, \dots, n\}$, the following holds: The matchings in $CM(\sigma^k(x))$ but not $M(\sigma^k(x))$ correspond to matchings (a,b) in $CM(x)$ such that $k \in (a,b)^*$.*

Proof. $M(x)$ consists of matchings (a,b) in $CM(x)$ such that $0 \notin (a,b)^*$. By rotating x, we see that the matchings in $M(\sigma^k(x))$ correspond to matchings (a,b) in $CM(x)$ such that $k \notin (a,b)*$. □

We say that the matchings that are in $CM(\sigma^k(x))$ but not $M(\sigma^k(x))$ are "cut" by the rotation of x that starts with the zero or one in this position k. The next proposition states that the elements of M_n are in fact the rotations that "cut" the most circular matchings. See Fig. 11 for examples of these cuts.

Proposition 13. *Let $X \in N_n$. For any representative $x \in B_n$ of X, the following holds. Let $k \in \{0, \dots, n-1\}$ be such that the number of matchings (a,b) in $CM(x)$ such that $k \in (a,b)^*$ is maximized. Then, $\sigma^k(x) \in M_n$.*

Proof. It is simple to see that

$$\big|U_1\big(\sigma^k(x)\big)\big| = \big|CU_1\big(\sigma^k(x)\big)\big| + \big|CM\big(\sigma^k(x)\big)\big| - \big|M\big(\sigma^k(x)\big)\big|.$$

Since the first term of the sum is fixed under rotation, $|U_1(\sigma^k(x))|$ is maximized when $|CM(\sigma^k(x))| - |M(\sigma^k(x))|$ is maximized. This quantity, by Proposition 12, is just the number of matchings (a,b) in $CM(x)$ such that $k \in (a,b)^*$. This is maximal by assumption. □

Proposition 14. *Let $x \in B_n$ with $|x| = k$. Then,*

$$CM(x) = CM\big(\tau^{n-2k}(x)\big).$$

In fact, if $x_1 < x_2 < \cdots < x_j$ is a chain in the Greene–Kleitman SCD, then

$$CM(x_1) = CM(x_j) \subset CM(x_2) = CM(x_{j-1}) \subset \cdots \subset CM(x_{\lceil \frac{j}{2} \rceil}) = CM(x_{\lceil \frac{j+1}{2} \rceil}).$$

Proof. We claim that, in the poset of circular matchings, there are no $(a,b) \in CM(x) \setminus M(x)$, $(c,d) \in M(x)$, with $(a,b) <_m (c,d)$. To demonstrate the claim, suppose that such a pair of matchings exists, and note that $0 \in (a,b)^*$, because $b < a$. Also, note that $0 \notin (c,d)^*$, because $c < d$. So, $(a,b)^* \nsubseteq (c,d)^*$, which implies that $(a,b) \nleq_m (c,d)$. By Proposition 9, since n is in each $(a,b)^*$ in $CM(x) \setminus M(x)$, $CM(x) \setminus M(x)$ is totally ordered.

By properties of the Greene–Kleitman SCD of B_n, we know that $M(x) = M(\tau^{n-2k}(x))$. So, it is enough to show that $CM(x) \setminus M(x) = CM(\tau^{n-2k}(x)) \setminus M(\tau^{n-2k}(x))$. Note that while $|y| < n/2$, $CM(y) \subseteq CM(\tau(y))$. If $|y| \geqslant n/2$, then $CM(y) \supset CM(\tau(y))$. During each step in the circular matching process, the leftmost circularly unmatched one is paired to the rightmost circularly unmatched zero. If (a,b) is a circular matching made earlier in the circular matching process than (c,d), then b is to the left of d, and a is to the right of c. In other words, $b < d$ and $c < a$. So, $(a,b) <_m (c,d)$. If $|y| < n/2$, then there are more zeros than ones, so all of the ones are circularly matched. The one added by $\tau(y)$ is to the right of all of the circularly matched ones in $U_1(y)$, so if it is circularly matched, it will be circularly matched last. (Since $|y| < n/2$, the zero we changed was not circularly matched, and this new one will not affect any of the circular matchings already present in $CM(y)$.) In other words, the new circular matching (if any) made with this new one will be the greatest element in the chain of matchings in $CM(\tau(y)) \setminus M(\tau(y))$.

Now, we assume $|y| \geqslant n/2$. In this case, all of the zeros are circularly matched. So, when we apply τ, we change the leftmost (smallest) element of $U_0(y)$ to a one. This zero was circularly matched, so we are removing a circular matching. But, because the zero was the leftmost, the circular matching we remove is the maximal matching in the chain of matchings in $CM(y) \setminus M(y)$. □

5. Three lemmas

In this section, we will use the properties of circular matching to prove the lemmas.

Proof of Lemma 6. Note that for the first part of the lemma, it is enough to show that if $x \in M_n$ with $|x| = k < n/2$, then $\tau(x) \in M_n$. Let x be as above. Then we know that $M(x) = M(\tau(x))$. By changing a zero to a one, at most one circular matching can be added. By Proposition 14, if k is not a middle level, then $CM(x) \subset CM(\tau(x))$. So $CM(\tau(x)) \setminus M(\tau(x))$ has one more circular matching than $CM(x) \setminus M(x)$. Thus, by Proposition 10 $\tau(x)$ also has the maximum cardinality of $CM(\tau(x)) \setminus M(\tau(x))$ over all rotations of $\tau(x)$. Thus, by Propositions 12 and 13, $\tau(x) \in M_n$. If $k < n/2$ is a middle level, then by Proposition 14, $CM(x) = CM(\tau(x))$, so by the same reasoning as above, $\tau(x) \in M_n$. This completes the proof of the first part of the lemma.

Now, suppose $|x| = k > n/2$. By Proposition 14, $CM(x) = CM(\tau^{n-2k}(x))$. So, $\tau^{n-2k}(x) \in M_n$ and $|\tau^{n-2k}(x)| < n/2$. So, by the first part of the lemma we have already proved, $\tau(\tau^{n-2k}(x)) - \tau^{n-2k+1}(x) \in M_n$. But by applying Proposition 14 again, $CM(\tau^{n-2k+1}(x)) = CM(\tau^{-1}(x))$. Thus, $\tau^{-1}(x) \in M_n$. □

Proof of Lemma 7. Let x, y be as in the statement of the lemma with $|x| \geqslant n/2$ and $y = \sigma^k(x)$. By Proposition 14, $\tau(x)$ and $\tau(y)$ are obtained from x and y, respectively by changing the zero in the maximal matching in $CM(x) \setminus M(x)$ and $CM(y) \setminus M(y)$ to a one. Let (a,b) be the maximal matching in $CM(x) \setminus M(x)$. By Proposition 10, $CM(y)$ is a rotation of $CM(x)$. First assume that $(a+k, b+k) \in CM(y) \setminus M(y)$. If $(a+k, b+k)$ is not maximal in $CM(y) \setminus M(y)$, then there was some matching in $M(x)$ that covered (a,b). We saw in the proof of Proposition 14 that this isn't possible. So, $(a+k, b+k)$ is maximal in $CM(y) \setminus M(y)$. Then, $(a+k, b+k)$ is the matching removed by $\tau(y)$. Thus, $CM(\tau(y))$ is a rotation of $CM(\tau(x))$, which implies that $\tau(x) \sim \tau(y)$. Next, assume that $(a+k, b+k) \in M(y)$. Then, if (c,d) is another matching in $CM(x) \setminus M(x)$, $(c,d) \subset (a,b)$ means that $(c+k, d+k) \subset (a+k, b+k)$. Therefore, $(c+k, d+k) \in M(y)$. Essentially, this means that the set of circular matchings cut by x is disjoint from the set of circular matchings cut by y. Note that since x and y are both in M_n, and they have the same number of ones, $|CM(x) \setminus M(x)| = |CM(y) \setminus M(y)|$. Then,

$$\begin{aligned}\bigl|CM\bigl(\tau(x)\bigr) \setminus M\bigl(\tau(x)\bigr)\bigr| &= \bigl|\bigl(CM(x) \setminus M(x)\bigr) \setminus \bigl\{(a,b)\bigr\}\bigr| \\ &= \bigl|CM(x) \setminus M(x)\bigr| - 1 \\ &= \bigl|CM(y) \setminus M(y)\bigr| - 1. \end{aligned} \tag{1}$$

On the other hand,

$$\bigl|CM\bigl(\sigma^k\bigl(\tau(x)\bigr)\bigr) \setminus M\bigl(\sigma^k\bigl(\tau(x)\bigr)\bigr)\bigr| = \bigl|CM(y) \setminus M(y)\bigr|. \tag{2}$$

So, $\tau(x) \notin M_n$.

In a symmetrical argument, we also get that $\tau(y) \notin M_n$. $\square$

Proof of Lemma 8. Since $x \sim y$, by Proposition 10, $CM(x)$ is a rotation of $CM(y)$. By Proposition 14, $CM(x) = CM(\tau^{n-2k}(x))$ and $CM(y) = CM(\tau^{n-2k}(y))$. So, $CM(\tau^{n-2k}(x))$ is a rotation of $CM(\tau^{n-2k}(y))$. Since $|\tau^{n-2k}(x)| = |\tau^{n-2k}(y)| > n/2$, all of the circularly unmatched positions are ones. Thus, $\tau^{n-2k}(x) \sim \tau^{n-2k}(y)$. $\square$

The proofs of the lemmas complete the proof of Theorem 5.

6. Additional properties and related conjectures

A motivating application for finding symmetric chain decompositions for N_n is related to finding symmetric Venn diagrams.

Definition 15. An *independent family* is a collection of n curves in the plane such that every subset of $[n]$ is represented at least once in the regions formed by the intersections of the interiors of the curves. A *Venn diagram* is an independent family where each subset is represented exactly once [10].

Definition 16. A *rotationally symmetric independent family* is an independent family of n congruent curves such that each curve is a rotation of the other curves by some multiple of $2\pi/n$ radians about a fixed point. A *rotationally symmetric Venn diagram* is a rotationally symmetric independent family that is also a Venn diagram [6].

Grünbaum [5] proves that any independent family of n curves must have at least $2 + n(|N_n| - 2)$ regions. He also shows that rotationally symmetric independent families of n curves exist for all n. He asks if a rotationally symmetric independent family of n curves with $2 + n(|N_n| - 2)$ regions can be found for each n.

Griggs, Killian, and Savage show in [7] that rotationally symmetric Venn diagrams of p curves exist when p is prime. It is simple to see that for prime p, any Venn diagram has the minimum number of regions. That is, the number of regions is $|B_p|$, which is equal to $2 + p(|N_p| - 2)$. In order to prove that these Venn diagrams exist, this method required the existence of an SCD for N_p with an additional property, defined below.

Definition 17. Let $starter(C)$ be the element of minimum rank in the chain C, and let $terminator(C)$ be the element of maximum rank in the chain C. We say that chain C^* *covers* chain C if there is an element $x \in C^*$ such that $starter(C)$ covers x and an element $y \in C^*$ such that y covers $terminator(C)$. Let A be an SCD in a finite ranked poset. A has the *chain cover property* if each chain in A that is not of maximal length is covered by some other chain in A. [7].

Jiang proved that, given an SCD with the chain cover property for N_n, there exists a rotationally symmetric independent family of n curves, using the same methods as [7].

Theorem 18. *(See Jiang [9].) Let R_n, a subposet of B_n, be a complete set of representatives of the elements of N_n such that each necklace element is represented exactly once. If there exists an SCD of R_n with the chain cover* *property, then there exists a rotationally symmetric independent family of n curves, with number of regions that reaches the lower bound,* $2 + n(|N_n| - 2)$.

By being slightly more specific about which representatives we delete, we can construct an SCD for N_n that has the chain cover property. By the theorem above, this will give us a rotationally symmetric independent family of n curves with $2 + n(|N_n| - 2)$ regions. This settles Grünbaum's question in [5].

Theorem 19. *For all n,* N_n *has an SCD with the chain cover property.*

Proof. First, we show that the Greene–Kleitman SCD for B_n, restricted to M_n, has the chain cover property. Let C be a nonempty chain in the Greene–Kleitman SCD for B_n, restricted to M_n. Note that if a chain is not shortened when we restrict it to M_n, then the element of smallest rank has no unmatched ones. Unless the element consists of all zeros, there is some rotation of it with at least one unmatched one. Therefore, the only unmodified chain is the chain beginning with $(0, 0, \dots, 0)$. This is the longest chain in the SCD, and it doesn't need to covered by any other chain. Now, we can assume that C was shortened when we restricted it to M_n. Let $x = \mathit{starter}(C)$, and $y = \mathit{terminator}(C)$. Then, $\tau^{-1}(x) \notin M_n$ and $\tau(y) \notin M_n$. So then, some rotation $\sigma^k(\tau(y))$ is in a longer chain in the SCD restricted to M_n. By Lemma 8, $\sigma^k(\tau^{-1}(x))$ is in the same chain. So, C is covered by this longer chain. Therefore, the Greene–Kleitman SCD for B_n, restricted to M_n, has an SCD with the chain cover property.

Next, we iteratively remove duplicate representatives in a way that preserves the following properties: The resulting chains form symmetric chains satisfying the chain cover property, and, for all $x \in N_n$, all chains containing a representative of x are of the same length. Lemmas 6, 7, and 8 show that the SCD for M_n satisfies both properties. For each iteration, we choose a chain C that contains an element of N_n that is duplicated in at least one other chain. Then, we choose $x \in C$ with $|x| \leqslant n/2$ such that $|x|$ is closest to $n/2$ while having the property of being duplicated in another chain. Suppose that the chains $C_1, C_2, \dots, C_k$ are the other chains in the SCD of M_n that contain rotations of x. If $|x| = n/2$, then we simply delete the chains $C_1, C_2, \dots, C_k$. We are only deleting elements that are rotations of elements in C. So, the resulting SCD still contains at least one representative of each element of N_n, and it satisfies the properties above.

Now, assume that $|x| < n/2$. In this case, we delete the rotations of

$$\{x, \tau^{-1}(x), \tau^{-2}(x), \dots\} \cup \{\tau^{n-2k}(x), \tau^{n-2k+1}(x), \tau^{n-2k+2}(x), \dots\}$$

in the chains $C_1, C_2, \dots, C_k$. Call the shortened chains $C'_1, C'_2, \dots, C'_k$. Now, all of the elements of C are unique to the remaining SCD, so C will not be modified again. Each of $C'_1, C'_2, \dots, C'_k$ are covered by C, preserving the chain cover property.

If some element in a chain $C'_1, C'_2, \dots, C'_k$ is duplicated, it must have been in a chain originally the same length as $C_1, C_2, \dots, C_k$. Using Lemma 8 this means that some rotation of x was also in this chain. So, if any elements of $C'_1, C'_2, \dots, C'_k$ are not unique in the resulting SCD, they must be duplicated only in one or more chains in $C'_1, C'_2, \dots, C'_k$. So, any remaining duplicated elements remain in chains of equal length. Therefore, both of the above properties are preserved.

In each iteration, we reduce the number of duplicated elements of N_n. By iterating until there are no more duplicated elements, we get an SCD for N_n that has the chain cover property. □

The following corollary follows from Theorems 18 and 19.

Corollary 20. *For all n, there exists a rotationally symmetric independent family of n curves with* $2 + n(|N_n| - 2)$ *regions.*

7. Open questions

A quotient closely related to N_n is the "true necklace," meaning B_n/G, where G is the group of automorphisms that includes both rotations and inversions. Does the "true necklace" have an SCD? By Stanley [11], we know that B_n/G is also Peck. One approach to this problem would be to try to show that B_n/G has the LYM property or the normalized matching property. Another approach would be to define a new type of matching or structuring that allows one to prove Lemmas 6, 7, and 8 (or other similar lemmas) for this quotient of B_n. The "true necklace" is actually a quotient of N_n, which leads to a third approach. This strategy would involve starting with an SCD given in this paper for N_n and show that there is some method to remove the "extra" representatives of the elements of B_n/G.

Let G and H be two groups of automorphisms on B_n, and K the group of automorphisms generated by G and H. Then, if B_n/G and B_n/H are SCOs, is B_n/K also an SCO?

Are there other quotients of the Boolean lattice that have SCDs? Can we show that in general, any quotient of B_n is an SCO?

Instead of using the Boolean lattice, use the poset of subsets of a multiset under the rotation automorphism. This would correspond to strings with not only zeros and ones, but each position is filled by a number in $\{0, 1, \ldots, k\}$. Visually, these necklaces could have $k + 1$ different "colors" of beads.

Acknowledgments

I would like to thank Jerrold Griggs for suggesting this problem and allowing me to stubbornly stick with it, providing numerous related papers, and proofreading and commenting on countless drafts.

References

[1] Ian Anderson, Combinatorics of Finite Sets, Dover Publications, Inc., Mineola, NY, 2002.
[2] N.G. de Bruijn, Ca. van Ebbenhorst Tengbergen, D. Kruyswijk, On the set of divisors of a number, Nieuw Arch. Wiskunde 23 (1951) 191–193.
[3] Konrad Engel, Sperner Theory, Cambridge University Press, New York, NY, 1997.
[4] Curtis Greene, Daniel J. Kleitman, Strong versions of Sperner's theorem, J. Combin. Theory Ser. A 20 (1976) 80–88.
[5] Branko Grünbaum, The search for symmetric Venn diagrams, Geombinatorics 8 (1999) 104–109.
[6] Branko Grünbaum, Venn diagrams and independent families of sets, Math. Mag. 48 (1975) 12–23.
[7] Jerrold R. Griggs, Charles E. Killian, Carla D. Savage, Venn diagrams and symmetric chain decompositions in the Boolean lattice, Electron. J. Combin. 11 (2004).
[8] Jerrold R. Griggs, Sufficient conditions for a symmetric chain order, SIAM J. Appl. Math. 32 (1977) 807–809.
[9] Jiang Zongliang, Symmetric chain decompositions and independent families of curves, MS thesis, North Carolina State University, http://www.lib.ncsu.edu/theses/available/etd-07072003-035905/unrestricted/etd.pdf, 2003.
[10] Jiang Zongliang, Carla D. Savage, On the existence of symmetric chain decompositions in a quotient of the Boolean lattice, Discrete Math. 309 (2009) 5278–5283.
[11] Richard P. Stanley, Quotients of Peck posets, Order 1 (1984) 29–34.